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AN ANALYSIS OF SOME BODY FORCES AND SURFACE FORCES THAT ARE TOGETHER CONSERVATIVE

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Abstract—The bilinear tensor operation $A \times B$ corresponding to the quadratic operation $A \times A \equiv A^*$, and the tensor operation $A \# B \equiv Ae_i \times Be_i$, are introduced.

Body forces $\mathbf{b}(\mathbf{X}, \mathbf{Y}, F)$ which do path-independent work during all fixed-surface deformations are shown to have the form

$$\mathbf{b} = F^*\mathbf{p} + F\# q + \mathbf{r}$$

and to be associated with tractions

$$\mathbf{f} = (\pi F^* + 2Q \times F + R)\mathbf{N}$$

and volume potential densities

$$\Phi = \pi \det F + Q \cdot F^* + R \cdot F + \varphi.$$

The coefficients of $\mathbf{b}$ and Φ satisfy the equations

$$\mathbf{p} = \mathrm{div}_{\mathbf{Y}}\, Q^{\mathrm{T}} - \partial\pi/\partial\mathbf{X}$$
$$q = \mathrm{curl}_{\mathbf{Y}}\, R^{\mathrm{T}} + \mathrm{curl}_{\mathbf{X}}\, Q$$
$$\mathbf{r} = \partial\varphi/\partial\mathbf{Y} - \mathrm{div}_{\mathbf{X}}\, R.$$

Moreover, $\mathbf{f} = (\partial\Phi/\partial F)\mathbf{N}$ and $\mathbf{b} = \partial\Phi/\partial\mathbf{Y} - \mathrm{Div}_{\mathbf{X}}\,(\partial\Phi/\partial F)$.

The paper concludes with several generic examples.

1. INTRODUCTION

The configurations of a deformable body can be represented by a family of vector functions $\mathbf{Y} = \boldsymbol{\xi}(\mathbf{X})$ with a common domain D in E^3. One may think of D either as an initial configuration of the body or as the domain of parametrization of a system of curvilinear coordinates. In either case the coordinates $\mathbf{X}$ of the points in D serve as labels for the material points of the body. Each point in the deformed configuration $\boldsymbol{\xi}(D)$ has material coordinates $\mathbf{X}$ and spatial coordinates $\mathbf{Y} = \boldsymbol{\xi}(\mathbf{X})$. The map $\mathbf{Y} = \boldsymbol{\xi}(\mathbf{X})$ is one-to-one and the deformation gradient $F = \partial\mathbf{Y}/\partial\mathbf{X}$ has a positive determinant.

I have two main objects in this paper. One is to introduce some tensor operations which help simplify the use of vector and tensor methods; the other is to characterize those body forces $\mathbf{b}(\mathbf{X}, \mathbf{Y}, F)$ for which there exist surface forces $\mathbf{f}(\mathbf{X}, \mathbf{Y}, F)$ such that the two together are conservative, in the sense that the work they do along any configuration path depends only on the initial and final configurations of the body. $\mathbf{b}$ and $\mathbf{f}$ have the dimensions of force per unit volume of $\mathbf{D}$ and force per unit area of ∂D, respectively.

First, suppose that $\Phi(\mathbf{X}, \mathbf{Y}, F)$ is real valued, and that $\mathbf{f}$ and $\mathbf{b}$ are defined by

$$\mathbf{f} = (\partial\Phi/\partial F)\mathbf{N}$$
$$\mathbf{b} = \partial\Phi/\partial\mathbf{Y} - \mathrm{Div}_{\mathbf{X}}\,(\partial\Phi/\partial F) \equiv L\Phi,$$

where $\mathrm{Div}_{\mathbf{X}}\,(\partial\Phi/\partial F) \equiv \mathrm{div}_{\mathbf{X}}\,[(\partial\Phi/\partial F)(\mathbf{X}, \boldsymbol{\xi}(\mathbf{X}), F(\mathbf{X}))]$.

Then the work done by $\mathbf{b}$ and $\mathbf{f}$ together along any configuration path $\mathbf{Y} = \boldsymbol{\xi}(\mathbf{X}, \varepsilon)$, $0 \leqslant \varepsilon \leqslant 1$, is

$$\int_0^1 \left[\int_D (\partial \mathbf{Y}/\partial \varepsilon) \cdot \mathbf{b} \, \mathrm{d}V + \int_{\partial D} (\partial \mathbf{Y}/\partial \varepsilon) \cdot (\partial \Phi/\partial F)\mathbf{N} \, \mathrm{d}\Sigma \right] \mathrm{d}\varepsilon$$

$$= \int_0^1 \int_D \langle (\partial \mathbf{Y}/\partial \varepsilon) \cdot [\mathbf{b} + \mathrm{Div}_{\mathbf{X}} (\partial \Phi/\partial F)] + (\partial F/\partial \varepsilon) \cdot (\partial \Phi/\partial F) \rangle \, \mathrm{d}V \, \mathrm{d}\varepsilon$$

$$= \int_0^1 \int_D [(\partial \mathbf{Y}/\partial \varepsilon) \cdot (\partial \Phi/\partial \mathbf{Y}) + (\partial F/\partial \varepsilon) \cdot (\partial \Phi/\partial F)] \, \mathrm{d}V \, \mathrm{d}\varepsilon$$

$$= \int_D \left(\int_0^1 \partial \Phi/\partial \varepsilon \, \mathrm{d}\varepsilon \right) \mathrm{d}V$$

$$= \int_D \Phi_1(\mathbf{X}) \, \mathrm{d}V - \int_D \Phi_0(\mathbf{X}) \, \mathrm{d}V,$$

where $\Phi_\alpha(\mathbf{X}) = \Phi(\mathbf{X}, \boldsymbol{\zeta}(\mathbf{X}, \alpha), F(\mathbf{X}, \alpha))$.

The body force $L\Phi$ and the surface force $(\partial \Phi/\partial F)\mathbf{N}$ are together conservative and have the volume potential density Φ.

Conversely, if $\mathbf{b}'$ and $\mathbf{f}'$ are any body force and surface force together having the volume potential density Φ, then along any configuration path $\mathbf{Y} = \boldsymbol{\zeta}(\mathbf{X}, \varepsilon)$, $0 \leqslant \varepsilon \leqslant 1$,

$$\int_0^1 \left\langle \int_D (\partial \mathbf{Y}/\partial \varepsilon) \cdot (\mathbf{b}' - L\Phi) \, \mathrm{d}V + \int_{\partial D} (\partial \mathbf{Y}/\partial \varepsilon) \cdot [\mathbf{f}' - (\partial \Phi/\partial F)\mathbf{N}] \, \mathrm{d}\Sigma \right\rangle \mathrm{d}\varepsilon = 0$$

so $\mathbf{b}' = \mathbf{b} \equiv L\Phi$ and $\mathbf{f}' = \mathbf{f} \equiv (\partial \Phi/\partial F)\mathbf{N}$. (A computational proof for the case $\mathbf{b} = \mathbf{b}(\mathbf{X}, \mathbf{Y}, F)$ will be given in Section 3.)

Although $\mathbf{f}$ depends on $\mathbf{X}$, on $\mathbf{Y}$, and on $F \equiv \partial \mathbf{Y}/\partial \mathbf{X}$, $\mathbf{b}$ may depend in addition on the second derivatives of $\mathbf{Y}$ with respect to $\mathbf{X}$, through the term $\mathrm{Div}_{\mathbf{X}} (\partial \Phi/\partial F)$. However, in this paper attention will be restricted to body forces of the form $\mathbf{b}(\mathbf{X}, \mathbf{Y}, F)$. This class is large enough to contain the usual sort of body forces $\mathbf{b}(\mathbf{X}, \mathbf{Y})$, and also to allow hydrostatic pressures and dead loads, as well as elastic stress fields $\mathbf{f} \equiv S(\mathbf{X}, F)\mathbf{N}$ among the complementary boundary forces.

2. WHAT WILL BE PROVED

Suppose that $\mathbf{b}(\mathbf{X}, \mathbf{Y}, F)$ is a *conservable* body force, that is, a body force which does path-independent work within each set of configurations agreeing on ∂D, as would be the case if $\mathbf{b}$ together with some surface force were conservative. The following three theorems will be proved. Notation introduced in the statements of theorems is explained in the Appendix, where various algebraic lemmas to be used in Section 3 are also introduced.

Theorem 1. $\mathbf{b}$ *has the form* $\mathbf{b} = F^*\mathbf{p} + F \# q + \mathbf{r}$,

where $\mathrm{curl}_{\mathbf{X}}\, \mathbf{p} - \mathrm{div}_{\mathbf{Y}}\, q^{\mathrm{T}} = \mathbf{0}$

and $\mathrm{div}_{\mathbf{X}}\, q + \mathrm{curl}_{\mathbf{Y}}\, \mathbf{r} = \mathbf{0}$.

Comments. $\mathbf{p}$ and $\mathbf{r}$ are vector functions and q is a tensor function of $\mathbf{X}$ and $\mathbf{Y}$.

Theorem 2. *There is a surface force* $\mathbf{f} = (\pi F^* + 2Q \times F + R)\mathbf{N}$ *such that* $\mathbf{b}$ *and* $\mathbf{f}$ *are together conservative, with potential density* $\Phi = \pi \det F + Q \cdot F^* + R \cdot F + \varphi$. *The coefficients of* $\mathbf{b}$ *and* Φ *are related by the equations*

$$\mathbf{p} = \mathrm{div}_{\mathbf{Y}}\, Q^{\mathrm{T}} - \partial \pi/\partial \mathbf{X}$$

$$q = \mathrm{curl}_{\mathbf{Y}}\, R^{\mathrm{T}} + \mathrm{curl}_{\mathbf{X}}\, Q$$

$$\mathbf{r} = \partial \varphi/\partial \mathbf{Y} - \mathrm{div}_{\mathbf{X}}\, R.$$

Comments. π and φ are scalar functions and Q and R are tensor functions of **X** and **Y**. **f** is a variationally admissible traction in the sense of Edelen and Lagoudas (1986, pp. 665–666).

Theorem 3. *The body force* **b**, *the surface force* **f**, *and the potential density* Φ *are related by*

$$\mathbf{f} = (\partial\Phi/\partial F)\mathbf{N}$$

$$\mathbf{b} = \partial\Phi/\partial\mathbf{Y} - \mathrm{Div}_{\mathbf{X}}(\partial\Phi/\partial F) = L\Phi.$$

Comments. If **b** and some surface force **f**′ different from **f** have together the volume potential density Φ', then $L(\Phi'-\Phi) = \mathbf{b}-\mathbf{b} = \mathbf{0}$, so $\Phi'-\Phi \equiv \eta$ is a null Lagrangian. Edelen and Lagoudas (1986, p. 664) have shown that a null Lagrangian has the same form as Φ in Theorem 2; therefore so does $\Phi' = \Phi+\eta$, and therefore $\mathbf{f}' \equiv (\partial\Phi'/\partial F)\mathbf{N}$ has the same form as **f**.

The surface forces $(\partial\eta/\partial F)\mathbf{N}$ corresponding to null Lagrangians are those surface forces which are in themselves conservative. (See examples (b)–(e) in Section 4.)

3. PROOFS

Proof of Theorem 1

Suppose that $\delta_1\mathbf{Y}$ and $\delta_2\mathbf{Y}$ are variations of a configuration $\mathbf{Y} = \boldsymbol{\xi}(\mathbf{X})$ which vanish on ∂D, and suppose that $\mathbf{Y}(\cdot\,;\varepsilon_1,\varepsilon_2) \equiv \mathbf{Y}+\varepsilon_1\delta_1\mathbf{Y}+\varepsilon_2\delta_2\mathbf{Y}$ is a configuration of B for all sufficiently small ε_1 and ε_2, say for $(\varepsilon_1,\varepsilon_2)$ in some disk Δ centered at $(0,0)$.

Let $\mathbf{b}(\cdot\,;\varepsilon_1,\varepsilon_2)$ be the body force associated with the configuration $\mathbf{Y}(\cdot\,;\varepsilon_1,\varepsilon_2)$. Let $\varphi(\varepsilon_1,\varepsilon_2)$ be the work done by **b** along any path in Δ from $(0,0)$ to $(\varepsilon_1,\varepsilon_2)$. Then φ is well-defined, and its mixed partial derivatives are equal.

But

$$\begin{aligned}(\partial\varphi/\partial\varepsilon_1) &= \lim_{h\to 0}(1/h)[\varphi(\varepsilon_1+h,\varepsilon_2)-\varphi(\varepsilon_1,\varepsilon_2)]\\
&= \lim_{h\to 0}(1/h)\left[\int_{\varepsilon_1}^{\varepsilon_1+h}\int_D \mathbf{b}(\mathbf{X};\varepsilon,\varepsilon_2)\cdot(\partial/\partial\varepsilon)[\mathbf{Y}(\mathbf{X};\varepsilon,\varepsilon_2)]\,\mathrm{d}V\,\mathrm{d}\varepsilon\right]\\
&= \lim_{h\to 0}(1/h)\left[\int_{\varepsilon_1}^{\varepsilon_1+h}\int_D \mathbf{b}(\mathbf{X};\varepsilon,\varepsilon_2)\cdot\delta_1\mathbf{Y}\,\mathrm{d}V\,\mathrm{d}\varepsilon\right]\\
&= \int_D \mathbf{b}(\mathbf{X};\varepsilon_1,\varepsilon_2)\cdot\delta_1\mathbf{Y}\,\mathrm{d}V,\end{aligned}$$

and similarly $\partial\varphi/\partial\varepsilon_2 = \int_D \mathbf{b}(\mathbf{X};\varepsilon_1,\varepsilon_2)\cdot\delta_2\mathbf{Y}\,\mathrm{d}V$. Therefore

$$\begin{aligned}0 &= \partial^2\varphi/\partial\varepsilon_2\,\partial\varepsilon_1 - \partial^2\varphi/\partial\varepsilon_1\,\partial\varepsilon_2\\
&= (\partial/\partial\varepsilon_2)\left[\int_D \mathbf{b}(\mathbf{X};\varepsilon_1,\varepsilon_2)\cdot\delta_1\mathbf{Y}\,\mathrm{d}V\right] - (\partial/\partial\varepsilon_1)\left[\int_D \mathbf{b}(\mathbf{X};\varepsilon_1,\varepsilon_2)\cdot\delta_2\mathbf{Y}\,\mathrm{d}V\right]\\
&= \int_D [(\partial/\partial\varepsilon_2)\mathbf{b}(\mathbf{X};\varepsilon_1,\varepsilon_2)\cdot\delta_1\mathbf{Y} - (\partial/\partial\varepsilon_1)\mathbf{b}(\mathbf{X};\varepsilon_1,\varepsilon_2)\cdot\delta_2\mathbf{Y}]\,\mathrm{d}V.\end{aligned}$$

Since

$$(\partial/\partial\varepsilon_i)\mathbf{b}(\mathbf{X};\varepsilon_1,\varepsilon_2) = (\partial\mathbf{b}/\partial\mathbf{Y})(\partial\mathbf{Y}/\partial\varepsilon_i) + (\partial\mathbf{b}/\partial\mathbf{Y}_{,j})(\partial\mathbf{Y}_{,j}/\partial\varepsilon_i)$$
$$= (\partial\mathbf{b}/\partial\mathbf{Y})\delta_i\mathbf{Y} + (\partial\mathbf{b}/\partial\mathbf{Y}_{,j})\delta_i\mathbf{Y}_{,j},$$

it follows that

$$0 = \int_D \langle[(\partial\mathbf{b}/\partial\mathbf{Y})\delta_2\mathbf{Y} + (\partial\mathbf{b}/\partial\mathbf{Y}_{,j})\delta_2\mathbf{Y}_{,j}]\cdot\delta_1\mathbf{Y} - [(\partial\mathbf{b}/\partial\mathbf{Y})\delta_1\mathbf{Y} + (\partial\mathbf{b}/\partial\mathbf{Y}_{,j})\delta_1\mathbf{Y}_{,j}]\cdot\delta_2\mathbf{Y}\rangle\,\mathrm{d}V.$$

Judiciously integrating by parts (and writing $\mathbf{v}A$ for $A^{\mathrm{T}}\mathbf{v}$), we have

$$0 = \int_D \langle\delta_1\mathbf{Y}\,[(\partial\mathbf{b}/\partial\mathbf{Y}) - (\partial\mathbf{b}/\partial\mathbf{Y})^{\mathrm{T}} - (\partial\mathbf{b}/\partial\mathbf{Y}_{,j})_{,j}]$$
$$- \delta_1\mathbf{Y}_{,j}\,[(\partial\mathbf{b}/\partial\mathbf{Y}_{,j}) + (\partial\mathbf{b}/\partial\mathbf{Y}_{,j})^{\mathrm{T}}]\rangle\cdot\delta_2\mathbf{Y}\,\mathrm{d}V + \int_D [\delta_1\mathbf{Y}\,(\partial\mathbf{b}/\partial\mathbf{Y}_{,j})\cdot\delta_2\mathbf{Y}]_{,j}\,\mathrm{d}V.$$

The last integral vanishes, by the divergence theorem, since $\delta_1\mathbf{Y} = \delta_2\mathbf{Y} = \mathbf{0}$ on ∂D. Therefore

$$0 = \int_D \langle\delta_1\mathbf{Y}\,[(\partial\mathbf{b}/\partial\mathbf{Y}) - (\partial\mathbf{b}/\partial\mathbf{Y})^{\mathrm{T}} - (\partial\mathbf{b}/\partial\mathbf{Y}_{,j})_{,j}] - \delta_1\mathbf{Y}_{,j}[(\partial\mathbf{b}/\partial\mathbf{Y}_{,j}) + (\partial\mathbf{b}/\partial\mathbf{Y}_{,j})^{\mathrm{T}}]\rangle\cdot\delta_2\mathbf{Y}\,\mathrm{d}V.$$

If each configuration $\mathbf{Y}$ is sufficiently rich in small variations $\delta\mathbf{Y}$ vanishing on ∂D [cf. Edelen and Lagoudas, 1986], then

$$(\partial\mathbf{b}/\partial\mathbf{Y}_{,1}) + (\partial\mathbf{b}/\partial\mathbf{Y}_{,1})^{\mathrm{T}} = 0$$
$$(\partial\mathbf{b}/\partial\mathbf{Y}_{,2}) + (\partial\mathbf{b}/\partial\mathbf{Y}_{,2})^{\mathrm{T}} = 0$$
$$(\partial\mathbf{b}/\partial\mathbf{Y}_{,3}) + (\partial\mathbf{b}/\partial\mathbf{Y}_{,3})^{\mathrm{T}} = 0 \tag{1}$$
$$(\partial\mathbf{b}/\partial\mathbf{Y}) - (\partial\mathbf{b}/\partial\mathbf{Y})^{\mathrm{T}} = (\partial\mathbf{b}/\partial\mathbf{Y}_{,j})_{,j}. \tag{2}$$

These equations have been derived from the symmetry condition

$$0 = \partial^2\varphi/\partial\varepsilon_2\,\partial\varepsilon_1 - \partial^2\varphi/\partial\varepsilon_1\,\partial\varepsilon_2$$

in the same way that Euler–Lagrange equations are derived from a stationary condition of the form $\partial\varphi/\partial\varepsilon = 0$. Equations (1) imply that

$$\partial b^i/\partial Y^j_{,k} + \partial b^j/\partial Y^i_{,k} = 0, \quad i,j,k = 1,2,3.$$

It follows by brute force (see Fisher, 1987, Appendix 1, for a similar argument) that $\mathbf{b}$ can be written in the form

$$\mathbf{b} = p^i(\mathbf{Y}_{,i'} \times \mathbf{Y}_{,i''}) + (\mathbf{Y}_{,i} \times \mathbf{q}_i) + \mathbf{r}$$
$$= F^*\mathbf{p} + F \# q + \mathbf{r}, \tag{3}$$

where p^i, $\mathbf{q}_i$ and $\mathbf{r}$ are functions of $\mathbf{X}$ and $\mathbf{Y}$. Use of this representation of $\mathbf{b}$, and the component form of (2), produces the equations

$$(\partial_{j'}p^{j''} - \partial_{j''}p^{j'}) - \partial^k q^k_j = 0$$
$$-\partial_k q^i_k + (\partial^{i''}r^{i'} - \partial^{i'}r^{i''}) = 0$$

where $\partial_i = \partial/\partial X_i$ and $\partial^i = \partial/\partial Y^i$.

These equations have the vector form

$$\mathrm{curl}_{\mathbf{X}}\,\mathbf{p} - \mathrm{div}_{\mathbf{Y}}\,q^{\mathrm{T}} = \mathbf{0}$$
$$\mathrm{div}_{\mathbf{X}}\,q + \mathrm{curl}_{\mathbf{Y}}\,\mathbf{r} = \mathbf{0} \tag{4}$$

where $\mathbf{p}$ is the vector with components p^i and q is the tensor with components q^i_j.

Proof of Theorem 2

Let β be the 4-form

$$\beta \equiv -(p^j\,\mathrm{d}Y^1\,\mathrm{d}Y^2\,\mathrm{d}Y^3\,\mathrm{d}X_j + q^{i''}_{j''}\,\mathrm{d}Y^i\,\mathrm{d}Y^{i'}\,\mathrm{d}X_j\,\mathrm{d}X_{j'} + r^i\,\mathrm{d}Y^i\,\mathrm{d}X_1\,\mathrm{d}X_2\,\mathrm{d}X_3)$$

then

$$\begin{aligned} \mathrm{d}\beta &\equiv [(\partial_j p^{j'} - \partial_{j'} p^j) - \partial^k q^{k}_{j''}]\,\mathrm{d}Y^1\,\mathrm{d}Y^2\,\mathrm{d}Y^3\,\mathrm{d}X_j\,\mathrm{d}X_{j'} \\ &\quad + [-\partial_k q^{i''}_k + (\partial^{i'} r_i - \partial^i r_{i'})]\,\mathrm{d}Y^i\,\mathrm{d}Y^{i'}\,\mathrm{d}X_1\,\mathrm{d}X_2\,\mathrm{d}X_3 \\ &= 0, \end{aligned}$$

so there is a 3-form

$$\alpha \equiv -(\pi\,\mathrm{d}Y^1\,\mathrm{d}Y^2\,\mathrm{d}Y^3 + Q^j_i\,\mathrm{d}Y^{i'}\,\mathrm{d}Y^{i''}\,\mathrm{d}X_j + R^j_i\,\mathrm{d}Y^i\,\mathrm{d}X_{j'}\,\mathrm{d}X_{j''} + \varphi\,\mathrm{d}X_1\,\mathrm{d}X_2\,\mathrm{d}X_3)$$

such that

$$\begin{aligned} \mathrm{d}\alpha &\equiv (\partial_i \pi - \partial^k Q^i_k)\,\mathrm{d}Y^1\,\mathrm{d}Y^2\,\mathrm{d}Y^3\,\mathrm{d}X_i \\ &\quad + [(\partial_{j'} Q^j_{i''} - \partial_j Q^{j'}_{i''}) + (\partial^{i'} R^{j''}_i - \partial^i R^{j''}_{i'})]\,\mathrm{d}Y^i\,\mathrm{d}Y^{i'}\,\mathrm{d}X_j\,\mathrm{d}X_{j'} \\ &\quad + (\partial_k R^k_i - \partial^i \varphi)\,\mathrm{d}Y^i\,\mathrm{d}X_1\,\mathrm{d}X_2\,\mathrm{d}X_3 \\ &= \beta. \end{aligned}$$

The substitutions $\mathrm{d}Y^i = (\partial Y^i/\partial X_j)\,\mathrm{d}X_j + (\partial Y^i/\partial \varepsilon)\,\mathrm{d}\varepsilon$ produce

$$\begin{aligned} \beta &= [p^i(\mathbf{Y}_{,i'} \times \mathbf{Y}_{,i''}) + (\mathbf{Y}_{,i} \times \mathbf{q}_i) + \mathbf{r}] \cdot (\partial \mathbf{Y}/\partial \varepsilon)\,\mathrm{d}X_1\,\mathrm{d}X_2\,\mathrm{d}X_3\,\mathrm{d}\varepsilon \\ &= \mathbf{b} \cdot (\partial \mathbf{Y}/\partial \varepsilon)\,\mathrm{d}X_1\,\mathrm{d}X_2\,\mathrm{d}X_3\,\mathrm{d}\varepsilon, \\ \alpha &= -[\pi \det F + \mathbf{Q}_i \cdot (\mathbf{Y}_{,i'} \times \mathbf{Y}_{,i''}) + \mathbf{R}_i \cdot \mathbf{Y}_{,i} + \varphi]\,\mathrm{d}X_1\,\mathrm{d}X_2\,\mathrm{d}X_3 \\ &\quad - [\pi(\mathbf{Y}_{,j} \times \mathbf{Y}_{,j'}) + (\mathbf{Q}_j \times \mathbf{Y}_{,j'} - \mathbf{Q}_{j'} \times \mathbf{Y}_{,j}) + \mathbf{R}_{j''}] \cdot (\partial \mathbf{Y}/\partial \varepsilon)\,\mathrm{d}X_j\,\mathrm{d}X_{j'}\,\mathrm{d}\varepsilon \\ &= -(\pi \det F + Q \cdot F^* + R \cdot F + \varphi)\,\mathrm{d}X_1\,\mathrm{d}X_2\,\mathrm{d}X_3 \\ &\quad - (\pi F^* + 2Q \times F + R)\mathbf{N}\,\mathrm{d}\Sigma \cdot (\partial \mathbf{Y}/\partial \varepsilon)\,\mathrm{d}\varepsilon, \end{aligned}$$

where $\mathrm{d}\Sigma$ is the element of surface area on ∂D.

By the generalized Stokes' Theorem,

$$\begin{aligned} &\int_0^1 \int_D \mathbf{b} \cdot (\partial \mathbf{Y}/\partial \varepsilon)\,\mathrm{d}X_1\,\mathrm{d}X_2\,\mathrm{d}X_3\,\mathrm{d}\varepsilon = \int_{D \times [0,1]} \beta \\ &= \int_{\partial(D \times [0,1])} \alpha \\ &= \int_{D \times \{1\}} (\pi \det F + Q \cdot F^* + R \cdot F + \varphi)\,\mathrm{d}X_1\,\mathrm{d}X_2\,\mathrm{d}X_3 \\ &\quad - \int_{D \times \{0\}} (\pi \det F + Q \cdot F^* + R \cdot F + \varphi)\,\mathrm{d}X_1\,\mathrm{d}X_2\,\mathrm{d}X_3 \\ &\quad - \int_{\partial D \times [0,1]} (\pi F^* + 2Q \times F + R)\mathbf{N}\,\mathrm{d}\Sigma \cdot (\partial \mathbf{Y}/\partial \varepsilon)\,\mathrm{d}\varepsilon. \end{aligned}$$

Therefore if the surface load

$$\mathbf{f} = (\pi F^* + 2Q \times F + R)\mathbf{N}, \text{ per unit area of } \partial D,$$

were added to the body force $\mathbf{b}$, the two together would be conservative and have the volume potential density

$$\Phi = \pi \det F + Q \cdot F^* + R \cdot F + \varphi.$$

Comparison of α and $d\beta$ shows that

$$\begin{aligned} p^i &= \partial^k Q^i_k - \partial_i \pi \\ q^i_j &= (\partial_{j'} Q^{j''}_i - \partial_{j''} Q^{j'}_i) + (\partial^{i'} R^j_{i''} - \partial^{i''} R^j_{i'}) \\ r^i &= \partial^i \varphi - \partial_k R^k_i \end{aligned}$$

or

$$\begin{aligned} \mathbf{p} &= \mathrm{div}_Y\, Q^{\mathrm{T}} - \partial\pi/\partial\mathbf{X} \\ q &= \mathrm{curl}_Y\, R^{\mathrm{T}} + \mathrm{curl}_X\, Q \\ \mathbf{r} &= \partial\varphi/\partial\mathbf{Y} - \mathrm{div}_X\, R. \end{aligned}$$

(The curl of a tensor field is another tensor field. Note that $\mathrm{div}_X F^* = \mathbf{0}$ and $\mathrm{curl}_X F = 0$.)

Proof of Theorem 3

One proof was sketched in the Introduction. Here is a different proof, the details of which are useful for constructing examples.

Straightforward computation shows that

$$\partial(\det F)/\partial F = F^*, \quad \partial(Q \cdot F^*)/\partial F = 2Q \times F, \quad \text{and} \quad \partial(R \cdot F)/\partial F = R.$$

Therefore $(\partial\Phi/\partial F)\mathbf{N} = \mathbf{f}$.

To show that $L(\Phi) = \mathbf{b}$, i.e. that

$$\begin{aligned} L(\pi \det F + Q \cdot F^* + R \cdot F + \varphi) &= F^*\mathbf{p} + q \# F + \mathbf{r} \\ &= F^*(\mathrm{div}_Y\, Q^{\mathrm{T}} - \partial\pi/\partial\mathbf{X}) - (\mathrm{curl}_Y\, R^{\mathrm{T}} + \mathrm{curl}_X\, Q) \# F + (\partial\varphi/\partial\mathbf{Y} - \mathrm{div}_X\, R), \end{aligned}$$

it will be sufficient to show that

(i) $L(\pi \det F) = -F^*(\partial\pi/\partial\mathbf{X})$

(ii) $L(Q \cdot F^*) = F^*(\mathrm{div}_Y\, Q^{\mathrm{T}}) - (\mathrm{curl}_X\, Q) \# F$

(iii) $L(R \cdot F) = -(\mathrm{curl}_Y\, R^{\mathrm{T}}) \# F - \mathrm{div}_X\, R$

(iv) $L(\varphi) = \partial\varphi/\partial\mathbf{Y}$.

Proof of (i). $L(\pi \det F) = \partial(\pi \det F)/\partial\mathbf{Y} - \mathrm{Div}_X(\pi F^*)$, and

$$\begin{aligned} \mathrm{Div}_X(\pi F^*) &= \partial(\pi \mathbf{F}_{i'} \times \mathbf{F}_{i''})/\partial X_i \\ &= [\partial\pi/\partial X_i + (\partial\pi/\partial\mathbf{Y}) \cdot (\partial\mathbf{Y}/\partial X_i)](\mathbf{F}_{i'} \times \mathbf{F}_{i''}) + \pi\, \mathrm{div}_X F^* \\ &= F^*(\partial\pi/\partial\mathbf{X}) + [(\partial\pi/\partial\mathbf{Y})\mathbf{F}_i](\mathbf{F}_{i'} \times \mathbf{F}_{i''}) \\ &= F^*(\partial\pi/\partial\mathbf{X}) + F^* F^{\mathrm{T}}(\partial\pi/\partial\mathbf{Y}) \\ &= F^*(\partial\pi/\partial\mathbf{X}) + (\det F)(\partial\pi/\partial\mathbf{Y}) \\ &= F^*(\partial\pi/\partial\mathbf{X}) + \partial(\pi \det F)/\partial\mathbf{Y}. \end{aligned}$$

Proof of (ii). Since $\partial(Q\cdot F^*)/\partial F = 2Q\times F$, (ii) is equivalent to

$$\partial(Q\cdot F^*)/\partial \mathbf{Y} - \mathrm{Div}_{\mathbf{X}}(2Q\times F) = F^*(\mathrm{div}_{\mathbf{Y}}\,Q^{\mathrm{T}}) - (\mathrm{curl}_{\mathbf{X}}\,Q) \# F, \text{ or}$$
$$\mathrm{Div}_{\mathbf{X}}\,(2Q\times F) = (\mathrm{curl}_{\mathbf{X}}\,Q) \# F + \partial(Q\cdot F^*)/\partial \mathbf{Y} - F^*(\mathrm{div}_{\mathbf{Y}}\,Q^{\mathrm{T}}).$$

But

$$\begin{aligned}\mathrm{Div}_{\mathbf{X}}\,(2Q\times F) &= \partial_i(\mathbf{Q}_{i'}\times\mathbf{F}_{i''} - \mathbf{Q}_{i''}\times\mathbf{F}_{i'})\\ &= [\partial_i\mathbf{Q}_{i'} + (\partial^j\mathbf{Q}_{i'})(\partial_i Y^j)]\times\mathbf{F}_{i''} + \mathbf{Q}_{i'}\times\partial_i\mathbf{F}_{i''}\\ &\quad - \langle[\partial_i\mathbf{Q}_{i''} + (\partial^j\mathbf{Q}_{i''})(\partial_i Y^j)]\times\mathbf{F}_{i'} + \mathbf{Q}_{i''}\times\partial_i F_{i'}\rangle\\ &= (\partial_i\mathbf{Q}_{i'})\times\mathbf{F}_{i''} - (\partial_i\mathbf{Q}_{i''})\times\mathbf{F}_{i'}\\ &\quad + \mathbf{Q}_{i'}\times(\partial_i\mathbf{F}_{i''}) - \mathbf{Q}_{i''}\times(\partial_i\mathbf{F}_{i'})\\ &\quad + (\partial^j\mathbf{Q}_{i'})(\partial_i Y^j)\times\mathbf{F}_{i''} - (\partial^j\mathbf{Q}_{i''})(\partial_i Y^j)\times\mathbf{F}_{i'}.\end{aligned}$$

The first line of this expression is equal to:

$$\begin{aligned}&(\partial_{i'}\mathbf{Q}_{i''})\times\mathbf{F}_i - (\partial_{i''}\mathbf{Q}_{i'})\times\mathbf{F}_i\\ &= (\partial_{i'}\mathbf{Q}_{i''} - \partial_{i''}\mathbf{Q}_{i'})\times\mathbf{F}_i\\ &= (\mathrm{curl}_{\mathbf{X}}\,Q) \# F.\end{aligned}$$

The second line is equal to:

$$\begin{aligned}&\mathbf{Q}_{i'}\times(\partial_i\mathbf{F}_{i''}) - Q_{i'}\times(\partial_{i''}\mathbf{F}_i)\\ &= \mathbf{Q}_{i'}\times(\partial_i\mathbf{F}_{i''} - \partial_{i''}\mathbf{F}_i)\\ &= -\mathbf{Q} \# \mathrm{curl}_{\mathbf{X}}\,F\\ &= \mathbf{0}.\end{aligned}$$

The third line is equal to:

$$\begin{aligned}&[(\partial\mathbf{Q}_{i'}/\partial\mathbf{Y})\mathbf{F}_i]\times\mathbf{F}_{i''} - [(\partial\mathbf{Q}_{i''}/\partial\mathbf{Y})\mathbf{F}_i]\times\mathbf{F}_{i'}\\ &= [(\partial\mathbf{Q}_i/\partial\mathbf{Y})\mathbf{F}_{i''}]\times\mathbf{F}_{i'} - [(\partial\mathbf{Q}_i/\partial\mathbf{Y})\mathbf{F}_{i'}]\times\mathbf{F}_{i''}\\ &= 2[(\partial\mathbf{Q}_i/\partial\mathbf{Y})\times I](\mathbf{F}_{i''}\times\mathbf{F}_{i'}) \quad (\text{note}: 2A\times I = (\mathrm{trace}\,A)I - A^{\mathrm{T}})\\ &= [(\mathrm{div}_{\mathbf{Y}}\,\mathbf{Q}_i)I - (\partial\mathbf{Q}_i/\partial\mathbf{Y})^{\mathrm{T}}](\mathbf{F}_{i''}\times\mathbf{F}_{i'})\\ &= (\partial\mathbf{Q}_i/\partial\mathbf{Y})^{\mathrm{T}}(\mathbf{F}_{i'}\times\mathbf{F}_{i''}) - (\mathrm{div}_{\mathbf{Y}}\,\mathbf{Q}_i)(\mathbf{F}_{i'}\times\mathbf{F}_{i''})\\ &= \partial(Q\cdot F^*)/\partial\mathbf{Y} - F^*(\mathrm{div}_{\mathbf{Y}}\,Q^{\mathrm{T}}).\end{aligned}$$

Proof of (iii). Since $\partial(R\cdot F)/\partial F = R$, (iii) is equivalent to:

$$\partial(R\cdot F)/\partial\mathbf{Y} - \mathrm{Div}_{\mathbf{X}}\,R = -(\mathrm{curl}_{\mathbf{Y}}\,R^{\mathrm{T}}) \# F - \mathrm{div}_{\mathbf{X}}\,R, \text{ or}$$
$$\begin{aligned}\mathrm{Div}_{\mathbf{X}}\,R &= \mathrm{div}_{\mathbf{X}}\,R + \partial(R\cdot F)/\partial\mathbf{Y} + (\mathrm{curl}_{\mathbf{Y}}\,R^{\mathrm{T}}) \# F\\ &= \mathrm{div}_{\mathbf{X}}\,R + \partial(R_i\cdot F_i)/\partial\mathbf{Y} + (\mathrm{curl}_{\mathbf{Y}}\,\mathbf{R}_i)\times\mathbf{F}_i\\ &= \mathrm{div}_{\mathbf{X}}\,R + (\partial\mathbf{R}_i/\partial\mathbf{Y})^{\mathrm{T}}\mathbf{F}_i + (\mathrm{curl}_{\mathbf{Y}}\,\mathbf{R}_i)\times\mathbf{F}_i.\end{aligned}$$

But

$$\text{Div}_X\, R = (\partial \mathbf{R}_i/\partial X_i) + (\partial \mathbf{R}_i/\partial \mathbf{Y})(\partial \mathbf{Y}/\partial X_i)$$
$$= (\partial \mathbf{R}_i/\partial X_i) + (\partial \mathbf{R}_i/\partial \mathbf{Y})\mathbf{F}_i$$

and

$$(\partial \mathbf{R}_i/\partial \mathbf{Y})\mathbf{F}_i = (\partial \mathbf{R}_i/\partial \mathbf{Y})^{\mathrm{T}}\mathbf{F}_i + (\text{curl}_Y\, \mathbf{R}_i) \times \mathbf{F}_i,$$

since it is in general true that

$$(\text{curl}_Y\, \mathbf{a}) \times \mathbf{b} = (\partial \mathbf{a}/\partial \mathbf{Y})\mathbf{b} - (\partial \mathbf{a}/\partial \mathbf{Y})^{\mathrm{T}}\mathbf{b}.$$

Proof of (iv). Obvious.

4. EXAMPLES

(a) If $\gamma(\mathbf{Y})$ is gravitational potential, and $\rho(\mathbf{X})$ is mass density per unit volume of D, then $\mathbf{b} \equiv \rho(\mathbf{X})\partial\gamma/\partial\mathbf{Y}$ and $\mathbf{f} \equiv \mathbf{0}$ have together the potential density $\Phi = \rho(\mathbf{X})\gamma(\mathbf{Y})$.

Comment. It follows from the equations of Theorem 2 that only body forces of the form $\partial\varphi(\mathbf{X}, \mathbf{Y})/\partial\mathbf{Y}$ can do path-independent work by themselves, without the aid of any traction.

(b) It has been shown (Fisher, 1988) that any position-dependent pressure $\mathbf{f}\, d\Sigma \equiv \pi(\mathbf{Y})F^{*}\mathbf{N}\, d\Sigma = \pi(\mathbf{Y})\mathbf{n}\, d\sigma$ with $\pi = \text{div}_Y\, \mathbf{A}$ has the surface potential $\int \mathbf{A}(\mathbf{Y}) \cdot \mathbf{n}\, d\sigma$ and therefore the volume potential $\int \pi(\mathbf{Y})\, dv = \int_D \pi(\mathbf{Y}) \det F\, dV$. This formula $\Phi = \pi(\mathbf{Y}) \det F$ is now reconfirmed, since $\partial\Phi/\partial F = \pi(\mathbf{Y})F^{*}$, and by part (i) of the proof of Theorem 3, $L\Phi = \mathbf{0}$.

(c) A dead-load $\mathbf{f}(\mathbf{X})$ satisfying $\int_{\partial D} \mathbf{f}(\mathbf{X})\, d\Sigma = 0$ has the surface potential density $\mathbf{Y} \cdot \mathbf{f}$. To get its volume potential Φ, let $\mathbf{a}(\mathbf{X})$ be a vector field harmonic on D satisfying $(\partial\mathbf{a}/\partial\mathbf{X})\mathbf{N} = \mathbf{f}(\mathbf{X})$ on ∂D. Then $\Phi \equiv R \cdot F \equiv (\partial\mathbf{a}/\partial\mathbf{X}) \cdot F$ satisfies $\partial\Phi/\partial F = \partial\mathbf{a}/\partial\mathbf{X}$, so $\mathbf{f} = (\partial\Phi/\partial F)\mathbf{N}$; and by part (iii) of Theorem 3, $L\Phi = \mathbf{b} = \mathbf{0}$.

Comment. A non-zero dead-load of constant density with respect to the material coordinates does path-independent work but is not of the form $\mathbf{f} = (\pi F^{*} + 2Q \times F + R)\mathbf{N}$.

(d) Suppose $\mu(\mathbf{X})$ is a scalar field satisfying $\int_{\partial D} \mu(\mathbf{X})\, d\Sigma = 0$ and φ is a scalar field depending only on $\mathbf{Y}$. The surface force $\mathbf{f} = \mu(\mathbf{X})\partial\varphi/\partial\mathbf{Y}$ has the surface potential density $\mu(\mathbf{X})\varphi(\mathbf{Y})$, and if $\alpha(\mathbf{X})$ is any function harmonic in D satisfying $(\partial\alpha/\partial\mathbf{X}) \cdot \mathbf{N} = \mu$ on ∂D, then $\mathbf{f}$ together with the body force $\mathbf{b} \equiv \mathbf{0}$ has the volume potential density

$$\Phi = R \cdot F \equiv [(\partial\varphi/\partial\mathbf{Y}) \otimes (\partial\alpha/\partial\mathbf{X})] \cdot F = \partial\varphi/\partial\mathbf{Y} \cdot F(\partial\alpha/\partial\mathbf{X}).$$

Proof. $(\partial\Phi/\partial F)\mathbf{N} = R\mathbf{N} = \partial\varphi/\partial\mathbf{Y}\, [(\partial\alpha/\partial\mathbf{X}) \cdot \mathbf{N}] = \mu\, \partial\varphi/\partial\mathbf{Y} = \mathbf{f}$. Also $\text{curl}_Y\, R^{\mathrm{T}} = 0$ and $\text{div}_X\, R = \mathbf{0}$, and therefore by part (iii) of the proof of Theorem 3, $L\Phi \equiv \mathbf{b} = \mathbf{0}$.

(e) For any scalar field $\alpha(\mathbf{X})$ and vector field $\mathbf{a}(\mathbf{Y})$, let

$$\Phi = (\text{curl}_Y\, \mathbf{a}) \cdot F^{*}\partial\alpha/\partial\mathbf{X} = [(\text{curl}_Y\, \mathbf{a}) \otimes (\partial\alpha/\partial\mathbf{X})] \cdot F^{*} \equiv Q \cdot F^{*}.$$

Then $\text{div}_Y\, Q^{\mathrm{T}} = \mathbf{0}$ and $\text{curl}_X\, Q = 0$, so by (ii) of Theorem 3, $L\Phi = \mathbf{0}$. Therefore the surface force $\mathbf{f} = (\partial\Phi/\partial F)\mathbf{N} = (2Q \times F)\mathbf{N}$ is conservative, with volume potential density Φ.

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APPENDIX

Throughout this paper the summation convention is augmented by the successor relation $3 = 2' = 1''$, $2 = 1' = 3''$, $1 = 3' = 2''$. Summation is triggered only when there is an unprimed index in a cofactor.

For some arguments, it is easier to use components relative to a fixed orthonormal basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. In such a basis, $\mathbf{X} \equiv X_i\mathbf{e}_i$ and $\mathbf{A}_i \equiv A\mathbf{e}_i$. In particular, for the deformation gradient $F \equiv \partial\mathbf{Y}/\partial\mathbf{X}$, we have $\mathbf{F}_i = \partial\mathbf{Y}/\partial X_i = \mathbf{Y}_{,i}$.

The subscript $,i$ represents differentiation with respect to the material coordinate X_i.

∂_i and ∂^i are differentiation operators with respect to the material coordinate X_i and the position coordinate Y^i; that is, $\partial_i f = \partial f/\partial X_i$ and $\partial^i f = \partial f/\partial Y^i$.

The notation $\mathbf{v}A$ is used for $A^{\mathrm{T}}\mathbf{v}$.

If $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is any positively oriented orthonormal basis for E^3, i.e. any vector triple satisfying $\mathbf{e}_i = \mathbf{e}_{i'} \times \mathbf{e}_{i''}$, then the various tensor operations used in this paper can be defined in terms of the basis $\{\mathbf{e}_i\}$ by:

(i) $\operatorname{div} A$: $\mathbf{e}_i \cdot \operatorname{div} A = \operatorname{div}(\mathbf{e}_i A)$

(ii) $\operatorname{curl} A$: $\mathbf{e}_i \operatorname{curl} A = \operatorname{curl}(\mathbf{e}_i A)$

(iii) $A \cdot B$: $A \cdot B = A\mathbf{e}_i \cdot B\mathbf{e}_i$

(iv) $A \# B$: $A \# B = A\mathbf{e}_i \times B\mathbf{e}_i$

(v) A^*: $A^*\mathbf{e}_i = A\mathbf{e}_{i'} \times A\mathbf{e}_{i''}$

(vi) $A \times B$: $(A \times B)\mathbf{e}_i = (1/2)[A\mathbf{e}_{i'} \times B\mathbf{e}_{i''} - A\mathbf{e}_{i''} \times B\mathbf{e}_{i'}]$.

The operations div and curl are invariant under a rotational change of basis because the corresponding vector operations are invariant.

$A \cdot B = \operatorname{trace}(A^{\mathrm{T}}B)$ is also invariant. So is $A \# B$, since for any rotation O, $A\mathbf{e}_i \times B\mathbf{e}_i =$

$$(A^{\mathrm{T}}\mathbf{e}_{i'} \cdot B^{\mathrm{T}}\mathbf{e}_{i''} - A^{\mathrm{T}}\mathbf{e}_{i''} \cdot B^{\mathrm{T}}\mathbf{e}_{i'})\mathbf{e}_i =$$
$$(O^{\mathrm{T}}A^{\mathrm{T}}\mathbf{e}_{i'} \cdot O^{\mathrm{T}}B^{\mathrm{T}}\mathbf{e}_{i''} - O^{\mathrm{T}}A^{\mathrm{T}}\mathbf{e}_{i''} \cdot O^{\mathrm{T}}B^{\mathrm{T}}\mathbf{e}_{i'})\mathbf{e}_i = AO\mathbf{e}_i \times BO\mathbf{e}_i.$$

A^* and $A \times B$ are also rotation invariant, since $A^* = A^{-\mathrm{T}} \det A$ and $A \times B$ is bilinear in A and B, and commutative, and $A \times A = A^*$, so $A \times B = (1/2)[(A+B)^* - A^* - B^*]$.

Some further easy-to-prove algebraic properties:

$$A^*(\mathbf{a} \times \mathbf{b}) = A\mathbf{a} \times A\mathbf{b}$$

(therefore F^* transforms material surface area elements $\mathbf{N}\, d\Sigma$ into spatial surface area elements $\mathbf{n}\, d\sigma$)

$$A^* = \partial(\det A)/\partial A$$
$$\partial(B \cdot A)/\partial A = B$$
$$\partial(B \cdot A^*)/\partial A = 2B \times A$$
$$(A\mathbf{a}) \times \mathbf{b} - (A\mathbf{b}) \times \mathbf{a} = 2(A \times I)(\mathbf{a} \times \mathbf{b})$$
$$2(A \times I) = (\operatorname{trace} A)I - A^{\mathrm{T}}$$
$$(\operatorname{curl}_{\mathbf{Y}} \mathbf{a}) \times \mathbf{b} = (\partial\mathbf{a}/\partial\mathbf{Y})\mathbf{b} - (\partial\mathbf{a}/\partial\mathbf{Y})^{\mathrm{T}}\mathbf{b}.$$

$I \# A$ is the unique vector satisfying $(I \# A) \times \mathbf{v} = (A - A^{\mathrm{T}})\mathbf{v}$, for all vectors $\mathbf{v}$, so $I \# A = 0$ if and only if A is symmetric.

$$(A \# B) = A^*(I \# A^{-1}B) = -B^*(I \# B^{-1}A).$$

I omit proofs, since it is easier to prove these than to read through someone else's proof.